$\sin(a+b) = \sin a\cos b + \cos a \sin b$

$\cos(a+b) = \cos a\cos b + \sin a \sin b$

$\sin 2\theta = 2\sin\theta\cos\theta = (\sin\theta+\cos\theta)^2 - 1$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$ $\quad \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\frac{1}{\sin\theta} = \csc\theta$ $\quad \frac{1}{\cos\theta} = \sec\theta$ $\quad \frac{\cos\theta}{\sin\theta} = \cot\theta$

$$\int u(x)\, v'(x)\, dx = u(x)v(x) - \int v(x)\, u'(x)\, dx$$

$$\iint f(x)\cdot g(y)\, dx\, dy = \int f(x)\, dx \cdot \int g(y)\, dy$$

$\sinh(a+b) = \sinh a\cosh b + \cosh a \sinh b$

$\cosh(a+b) = \cosh a\cosh b + \sinh a\sinh b$

$\sinh 2\theta = 2\sinh\theta\cosh\theta$

$\cosh 2\theta = \cosh^2\theta + \sinh^2\theta$

$\sinh^2\theta = \frac{\cosh 2\theta - 1}{2}$ $\quad \cosh^2\theta = \frac{\cosh 2\theta + 1}{2}$

$\cosh^2\theta - \sinh^2\theta = 1$

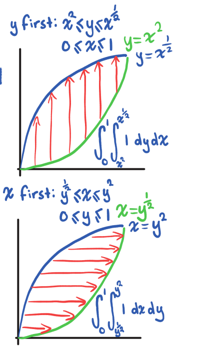
for a coordinate system change

$x = x(u,v),\ y = y(u,v)$

$$J(u,v) = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix} = \frac{\partial(x,y)}{\partial(u,v)}$$

So $dx\,dy = |J(u,v)|\,du\,dv$

$f(x)$	$f'(x)$	$F(x)$
x^n	nx^{n-1}	$\frac{1}{n}x^{n+1},\ x\neq -1$
x^{-1}		$\ln\lvert x\rvert$
a^{bx+c}	$b\ln(a)\, a^{bx+c}$	$(b\ln(a))^{-1} a^{bx+c}$
e^{ax+b}	ae^{ax+b}	$a^{-1}e^{ax+b}$
xe^x	$e^x + xe^x$	$(x-1)e^x$
$\sin x$	$\cos x$	$-\cos x$
$-\sin x$	$-\cos x$	$\cos x$
$\tan x$	$\sec^2 x$	$-\ln\lvert\cos x\rvert$
$\sinh x$	$\cosh x$	$\cosh x$
$\cosh x$	$\sinh x$	$\sinh x$
$\tanh x$	$\mathrm{sech}^2 x$	$\ln\lvert\cosh x\rvert$
$\frac{1}{1+x^2}$		$\arctan x$
$\frac{1}{1-x^2}$		$\frac{1}{2}\ln\left(\frac{\lvert 1-x\rvert}{\lvert 1+x\rvert}\right)$
$\frac{1}{\sqrt{1-x^2}}$		$\arcsin x$
$\frac{1}{\sqrt{1+x^2}}$		$\ln(x+\sqrt{1+x^2})$
$\frac{1}{\sqrt{x^2-1}}$		$\ln(x+\sqrt{x^2-1})$



$y = f(x)$

$\underset{\sim}{r}(t) = t\underset{\sim}{i} + f(t)\underset{\sim}{j}$

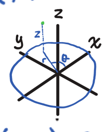
Polar

$x = r\cos\theta,\ y = r\sin\theta : J(r,\theta) = r$

Cylindrical

$r = \sqrt{x^2+y^2}$ $\quad z = z$

$\theta = \arctan\left(\frac{y}{x}\right)$ $x>0$



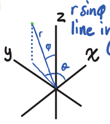
$x = r\cos\theta,\ y = r\sin\theta,\ z = z : J(r,\theta,z) = r$

Spherical

$r = \sqrt{x^2+y^2+z^2}$

$\varphi = \arccos\left(\frac{z}{r}\right)$

$\theta = \arctan\left(\frac{y}{x}\right)$ $x>0$

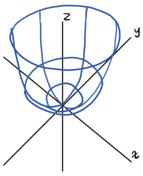


$r\sin\varphi$ is the radial line in xy-plane (r in polar & cylindrical)

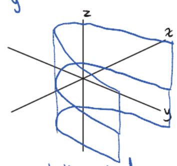
$x = r\sin\varphi\cos\theta,\ y = r\sin\varphi\sin\theta,\ z = r\cos\varphi : J(r,\varphi,\theta) = r^2\sin\varphi$

if one var^ble does not appear – generalised cylinder

$z = x^2 + y^2$

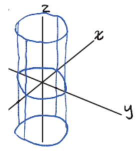


$y = x^2 - 1$

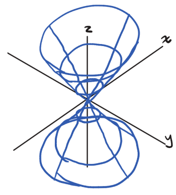


Parabolic cylinder

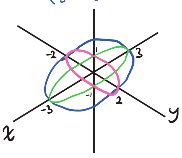
$x^2 + y^2 = 1$



$x^2 + y^2 = z^2$



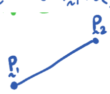
$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 + z^2 = 1$



$\left(\frac{x-3}{2}\right)^2 + (y-2)^2 = 1$

$\underset{\sim}{r}(t) = (2\cos t + 3)\underset{\sim}{i} + (\sin t + 2)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = (1-t)\underset{\sim}{P_1} + t(\underset{\sim}{P_2})$



for a solid V with density given by $p(x,y,z)$, the mass is given by $m = \iiint_V p(x,y,z)\, dx\,dy\,dz$

the moment of Inertia about a given axis (eg. x) is given by

$I_x = \iiint_V (y^2+z^2)\, p(x,y,z)\, dx\,dy\,dz$

the Centre of mass $(\bar{x},\bar{y},\bar{z})$ is given by

$\bar{x} = \frac{1}{m}\iiint_V x\, p(x,y,z)\, dx\,dy\,dz$

$\bar{y} = \frac{1}{m}\iiint_V y\, p(x,y,z)\, dx\,dy\,dz$

$\bar{z} = \frac{1}{m}\iiint_V z\, p(x,y,z)\, dx\,dy\,dz$

Potential Energy in a conservative vector field $\underset{\sim}{F}$

$PE = -f$

tangent vector $\underset{\sim}{r}(t)$ is by definition

$\frac{d\underset{\sim}{r}}{dt} = 2\underset{\sim}{i} + 3x\underset{\sim}{j}$

$\Rightarrow \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} = 2\underset{\sim}{i} + 3x\underset{\sim}{j}$

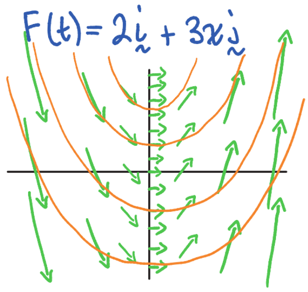
$\Rightarrow \frac{dx}{dt}\underset{\sim}{i} = 2\underset{\sim}{i}$, $\frac{dy}{dt}\underset{\sim}{j} = 3x\underset{\sim}{j}$

$\Rightarrow x(t) = 2t + x_0$, $\frac{dy}{dt} = 3(2t+x_0)$

$\Rightarrow y(t) = 3t^2 + 3x_0 t + y_0$

thus $\underset{\sim}{r}(t)$ is the family of curves given by $x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$

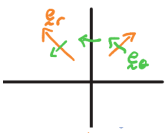
eg. $\underset{\sim}{F}(t) = 2\underset{\sim}{i} + 3x\underset{\sim}{j}$



The del Operator

$\nabla = \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}$

Polar



e_r direction of increasing radius

normalised $\{\frac{1}{r}e_\theta$ direction of increasing θ

$\nabla f = \frac{\partial f}{\partial r}e_r + \frac{\partial f}{\partial \theta}\frac{1}{r}e_\theta$

e_r and e_θ are analogous to $\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k}$ etc.

Cylindrical

$\nabla f = \frac{\partial f}{\partial r}e_r + \frac{\partial f}{\partial\theta}\frac{1}{r}e_\theta + \frac{\partial f}{\partial z}e_z$ $\quad \underset{\sim}{e_z} = \underset{\sim}{k}$

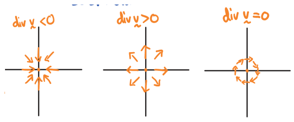
Spherical

$\nabla f = \frac{\partial f}{\partial r}e_r + \frac{\partial f}{\partial\varphi}\frac{1}{r}e_\varphi + \frac{\partial f}{\partial\theta}\frac{1}{r\sin\varphi}e_\theta$

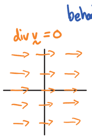
Divergence

$\nabla\cdot\underset{\sim}{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$= \left(\frac{d}{dx}\underset{\sim}{i} + \frac{d}{dy}\underset{\sim}{j} + \frac{d}{dz}\underset{\sim}{k}\right)\cdot(F_1\underset{\sim}{i} + F_2\underset{\sim}{j} + F_3\underset{\sim}{k})$



behaviour around 'sink' points



no 'sink' point

Cylindrical

for $\underset{\sim}{F}(r,\theta,z) = F_r\underset{\sim}{e_r} + F_\theta\underset{\sim}{e_\theta} + F_z\underset{\sim}{e_z}$

$\nabla\cdot f = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}$

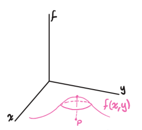
Spherical

$\nabla\cdot f = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\varphi}\frac{\partial F_\theta}{\partial\theta} + \frac{1}{r\sin\varphi}\frac{\partial}{\partial\varphi}(\sin\varphi F_\varphi)$

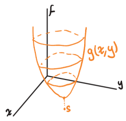
Laplacian

$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f$

$\nabla^2 f = \underbrace{\nabla\cdot(\overbrace{\nabla f}^{\text{Vector}})}_{\text{Scalar}}$ (divergence of the gradient)



$f(p)$ is greater than the average of surrounding points $\nabla^2 f|_p < 0$



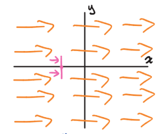
$f(s)$ is smaller than the average of surrounding points $\nabla^2 f|_s > 0$

Some times written $\int_C \underset{\sim}{F}\cdot d\underset{\sim}{r}$

Curl

$$\nabla\times\underset{\sim}{F} = \det\begin{bmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ F_1 & F_2 & F_3\end{bmatrix}$$

rotation $\sim$ curl $\neq 0$



no rotation $\sim$ curl $= 0$

Cylindrical

$\nabla\times\underset{\sim}{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}\right)e_r - \left(\frac{\partial F_z}{\partial r} - \frac{\partial F_r}{\partial z}\right)e_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(rF_\theta) - \frac{1}{r}\frac{\partial F_r}{\partial\theta}\right)e_z$

Spherical

$\nabla\times\underset{\sim}{F} = \left(\frac{1}{r\sin\varphi}\frac{\partial}{\partial\varphi}(\sin\varphi F_\theta) - \frac{1}{r\sin\varphi}\frac{\partial F_\varphi}{\partial\theta}\right)e_r - \left(\frac{1}{r}\frac{\partial}{\partial r}(rF_\theta) - \frac{1}{r\sin\varphi}\frac{\partial F_r}{\partial\theta}\right)e_\varphi + \left(\frac{1}{r}\frac{\partial}{\partial r}(rF_\varphi) - \frac{1}{r}\frac{\partial F_r}{\partial\varphi}\right)e_\theta$

Arc Length

$s = L(t) = \int_a^b\sqrt{\frac{d\underset{\sim}{r}}{dt}\cdot\frac{d\underset{\sim}{r}}{dt}}\,dt$

or $\int_b^a\left|\frac{d\underset{\sim}{r}}{dt}\right|dt$

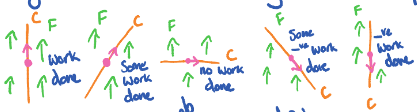
the line integral of a scalar field $f(x,y,z)$ along a curve C is given by

$$\int_C f(x,y,z)\,ds = \int_a^b f(\underset{\sim}{r}(t))\left|\frac{d\underset{\sim}{r}}{dt}\right|dt$$

the line integral does not depend on parametrisation or orientation

Line Integral of Vector Fields

think of the amount of work done on a particle moving along a curve C through a vector field F by the force represented by $\underset{\sim}{F}$



Some times written $W = \int_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$

if C is a closed curve $W = \oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$

also $W = \int_C P\,dx + Q\,dy + R\,dz$

$$W = \int_a^b \left(F(\underset{\sim}{r}(t)) \cdot \frac{d\underset{\sim}{r}}{dt} \right) dt$$

(direction) → reverse direction: $\int_{-C} \underset{\sim}{F} \cdot d\underset{\sim}{r} = -\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$

$$\nabla f = \frac{\partial f}{\partial x}\underset{\sim}{i} + \frac{\partial f}{\partial y}\underset{\sim}{j} + \frac{\partial f}{\partial z}\underset{\sim}{k}$$

$$\text{So } \nabla f \cdot \frac{d\underset{\sim}{r}}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \frac{df}{dt}$$

$$\text{So } \int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_{t=0}^{t=T} \frac{df}{dt}\, dt = \Big[f(x(t), y(t), z(t)) \Big]_0^T$$

therefore the integral only depends on the start & end points $\Rightarrow \underset{\sim}{F}$ is conservative

in this case f is called the potential

→ for $\underset{\sim}{F} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ where a, b, c are functions of x, y, z

$f = A' + B' + C' + D$ where A', B', C' are functions found by integrating the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ by considering partial functions $g_1(y,z), g_2(z)$

eg. $\underset{\sim}{F} = 2xz\underset{\sim}{i} + 2yz\underset{\sim}{j} + (x^2+y^2)\underset{\sim}{k}$

$\Rightarrow \frac{\partial f}{\partial x} = 2xz \Rightarrow f = zx^2 + g_1(y,z)$

$\therefore \frac{\partial f}{\partial y} = 0 + \frac{\partial g_1}{\partial y}$, but also: $\frac{\partial f}{\partial y} = 2yz$

$\Rightarrow 2yz = \frac{\partial g_1}{\partial y} \Rightarrow g_1 = zy^2 + g_2(z)$

$\Rightarrow f = zx^2 + zy^2 + g_2(z) \therefore \frac{\partial f}{\partial z} = x^2 + y^2 + g_2'(z)$

but also: $\frac{\partial f}{\partial z} = x^2 + y^2 \Rightarrow x^2 + y^2 = x^2 + y^2 + g_2'(z)$

$\Rightarrow g_2'(z) = 0 \Rightarrow g_2(z) = c_1$ (any constant)

$\Rightarrow f = zx^2 + zy^2 + c_1$ (let $c_1 = 0$)

then $\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = f(P_e) - f(P_s)$ where P_e and P_s are the start & end points of C

- if $\underset{\sim}{F}$ is conservative, there exists a potential (that is, $\underset{\sim}{F} = \nabla f$ for some scalar function f)
- if $\overset{\text{curl}}{\nabla \times \underset{\sim}{F}} = 0$, $\underset{\sim}{F}$ is conservative
  if $\underset{\sim}{F} = \nabla f \Rightarrow \nabla \times \underset{\sim}{F} = 0$

$$\nabla \times (\nabla f) = \det \begin{bmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} = \underset{\sim}{i}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right) - \underset{\sim}{j}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)\right) + \underset{\sim}{k}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)$$

if f is smooth, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ etc., so $\nabla \times \underset{\sim}{F} = 0$

a vector field $\underset{\sim}{F}$ is conservative if the line integral between any two points is path independent within a connected region D where $\underset{\sim}{F}$ is continuous

thus:

$$\int_{C_1} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_{C_2} \underset{\sim}{F} \cdot d\underset{\sim}{r} \quad \text{and} \quad \oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = 0$$

if $\underset{\sim}{F} = \nabla f$ for some scalar function f, then $\underset{\sim}{F}$ is conservative

$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_C \nabla f \frac{d\underset{\sim}{r}}{dt}\, dt = \int_{t=0}^{t=T} \nabla f \frac{d\underset{\sim}{r}}{dt}\, dt$$

Equivalent Statements

- $\underset{\sim}{F}$ is a conservative vector field
- $\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$ is independent of path C in D
- $\nabla \times \underset{\sim}{F} = 0$ at all points within D
- there exists a scalar field $f(x,y,z)$ such that $\nabla f = \underset{\sim}{F}$ for all x, y, z within D
- $\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = 0$ for any smooth closed curve C in D

Parametric Surfaces

Surfaces can be represented as:

a graph $\quad S = f$
$\quad z = f(x,y)$

a level surface $S = \{(x,y,z) : g(x,y,z) = 0\}$

Parametrically $\underset{\sim}{r}(u,v) = x(u,v)\underset{\sim}{i} + y(u,v)\underset{\sim}{j} + z(u,v)\underset{\sim}{k}$

$u, v \in R \subseteq \mathbb{R}^2$

tangent plane

the tangent plane at any point is the linear span of the tangent vectors $\frac{d\underset{\sim}{r}}{du}$ and $\frac{d\underset{\sim}{r}}{dv}$ at that point

$$\frac{d\underset{\sim}{r}}{du} = \frac{\partial x}{\partial u}\underset{\sim}{i} + \frac{\partial y}{\partial u}\underset{\sim}{j} + \frac{\partial z}{\partial u}\underset{\sim}{k}$$

$$\frac{d\underset{\sim}{r}}{dv} = \frac{\partial x}{\partial v}\underset{\sim}{i} + \frac{\partial y}{\partial v}\underset{\sim}{j} + \frac{\partial z}{\partial v}\underset{\sim}{k}$$

these depend on parametrisation, tangent plane does not! linear combination of all tangent vectors

normal vector

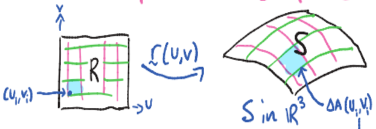
vector orthogonal to the tangent plane

$\Rightarrow$ cross product of tangent vectors

unit normal $\underset{\sim}{n} = \frac{\underset{\sim}{N}}{|\underset{\sim}{N}|}$ $\quad \underset{\sim}{N}$ depends on parametrisation

$\underset{\sim}{n}$ is independent (sign is dependent)

Area of Curved Surface



$\underset{\sim}{r}(u,v) = x(u,v)\underset{\sim}{i} + y(u,v)\underset{\sim}{j} + z(u,v)\underset{\sim}{k}$

take (u_i, v_i) points in the lattice

total area of $S \approx \sum_i \Delta A(u_i, v_i)$

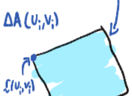
$\approx \sum_i \left| \frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v} \right| \Delta u \Delta v$

let $\Delta u, \Delta v \to 0$

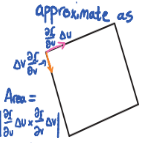
$= \iint_{(u,v) \in R} \left| \frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v} \right| du\,dv$

$=$ Area of S

$\left| \frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v} \right|$ is the Jacobian



approximate as



Area $= \left| \frac{\partial \underset{\sim}{r}}{\partial u}\Delta u \times \frac{\partial \underset{\sim}{r}}{\partial v}\Delta v \right|$

$\approx \left| \frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v} \right| \Delta u \Delta v$ at u_i, v_i

Surface Integral of Scalar Fields

the surface integral of a scalar field $f(x,y,z)$ across a curved surface S is given by

$\iint_S f(x,y,z)\, dS$ where dS is the scalar surface element

for $S = \underset{\sim}{r}(u,v)$, $u, v \in \mathbb{R}$

$\iint_R f\, |J(u,v)|\, du\,dv$ where $J(u,v)$ is the Jacobian and R is the surface in $\mathbb{R}^2$ mapped to S by $\underset{\sim}{r}(u,v)$ bounded by values for u & v

ie. $\iint_R = \int_{v=v_0}^{v=V} \int_{u=u_0}^{u=U}$ and f is $f(x,y,z)$ in terms of u & v

Flux through an open Surface

flux integral of $\underset{\sim}{F}$ through S with unit normal $\underset{\sim}{n}$

total flux $= \iint_S \underbrace{\underset{\sim}{F} \cdot \underset{\sim}{n}}_{\text{scalar}}\, dS$

for $S = \underset{\sim}{r}(u,v)$, $(u,v) \in R \subseteq \mathbb{R}^2$

total flux $= \iint_R (\underset{\sim}{F} \cdot \underset{\sim}{n})\, |J(u,v)|\, du\,dv$

then same as any surface integral of scalar field

however: $J(u,v) = \underset{\sim}{N}$

So $|J(u,v)| = |\underset{\sim}{N}|$

$$\text{So flux} = \iint_R (\underset{\sim}{F} \cdot \underset{\sim}{n})\, |J(u,v)|\, du\,dv = \iint_R \frac{(\underset{\sim}{F} \cdot \underset{\sim}{N})}{|\underset{\sim}{N}|} |\underset{\sim}{N}|\, du\,dv$$

$$= \iint_R \underset{\sim}{F} \cdot \underset{\sim}{N}\, du\,dv \quad \text{(get } \underset{\sim}{F} \text{ in terms of } u,v \text{ according to parametrisation)}$$

So we don't even need to calculate Jacobian

Flux through a closed Surface

Orientation matters!

We use outward normal vector

ie. if we get $\frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v} = \underset{\sim}{N}(u,v)$ that points inside, instead say $\frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v} = -\underset{\sim}{N}(u,v)$

We can split closed surfaces into smooth open surfaces and parametrise seperately → find sum of flux

the amount of vector field that passes through a surface

Proportional to projection of $\underset{\sim}{F}$ onto the normal to the surface

S_1 has higher flux than S_2

The Divergence Theorem

for a closed surface S enclosing a volume V, if $\underset{\sim}{F}$ is a 3D vector field

$$\underbrace{\iint_S (\underset{\sim}{F} \cdot \hat{\underset{\sim}{n}})\, dS}_{\text{Flux}} = \underbrace{\iiint_V \nabla \cdot \underset{\sim}{F}\, dV}_{\text{Divergence}}$$

where $\hat{\underset{\sim}{n}}$ is an outward unit normal to S

Conditions:

- S is a closed piecewise-smooth (it is made up of finitely many smooth surfaces (cube)) → all partial derivatives of all orders of $\underset{\sim}{r}$ exist and are continuous
  orientable surface → we can choose a continuous unit normal (always true for us) → non orientable: Mobius strip, Klein bottle
- $\underset{\sim}{F}$ has components with continuous first order partial derivatives → watch out for $\frac{1}{x}$ or $\frac{1}{x+y^2}$

Stoke's Theorem

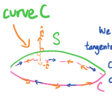
for a surface S bounded by a curve C in a vector field $\underset{\sim}{F}$

$$\underbrace{\iint_S \underbrace{(\nabla \times \underset{\sim}{F})}_{\text{curl}} \cdot \underbrace{\hat{\underset{\sim}{n}}}_{\text{unit norm to S}}\, dS}_{\text{integral of curl of } \underset{\sim}{F} \text{ over S}} = \underbrace{\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r}}_{\text{integral of } \underset{\sim}{F} \text{ over the boundary C}}$$

→ C must be a closed loop to be a boundary

$$\underbrace{\oint_C \left(P(x,y)\frac{dx}{dt} + Q(x,y)\frac{dy}{dt} \right) dt}_{\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r}} = \underbrace{\iint_R \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)}_{(\nabla \times \underset{\sim}{F}) \cdot \hat{\underset{\sim}{n}}} dA}_{\text{Area integral over R}}$$

line Integral over curve C



We must choose $\hat{\underset{\sim}{n}}$ and tangents in a compatible way called a positive orientation on C

- if you walk around C with your head pointing towards the normal, then your left hand touches S when you are going in the correct direction
- RHR - thumb to $\hat{\underset{\sim}{n}}_s$ (perpendicular to C), rotational path along curve of fingers
- RHR - thumb along tangent direction, fingers curl towards normal